Moment Generating Functions of Some Continuous Distributions

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ABSTRACT

In this paper, moment generating functions of some continuous distributions, viz., Student's t. F, Pareto, Noncentral t, Noncentral F, Beta and Lognormal distributions have been developed in a very simple forms from which moments of these distributions can be derived easily.

1. Introduction

Moment generating function (mgf) is a very important tool for finding the moments of a distribution. So far statisticians derived the mgfs of many distributions to find the moments of those distributions. But in the case of some distributions, they argued that the mgfs of them do not exist although their r-th raw moments exist. Mood, Graybill

and Boes (1974) found that in the case of Student's t, F and Pareto distributions the mgfs do not exist. The same comments had been given by Patel, Kapadia and Owen (1976). In that case the r-th raw moments had been derived from the expression $E(X^r)$. In this paper, we showed that the mgfs of Student's t, F and Pareto distributions also exist and from them the r-th raw moments had been calculated. The mgfs of noncentral t and noncentral F-distributions had also been derived.
Johnson and Kotz (1970) derived the mgf of beta distribution in such a complicated form that the moments of this distribution cannot be derived easily from that mgf. Here we showed the mgf of beta distribution from which the moments can be derived easily.
In the case of lognormal distribution Heyde (1963) said that the distribution could not be defined by its moments. Here we showed the mgf of lognormal distribution and its r-th raw moment had been derived directly from that mgf.

2. Main results

To find the mgfs of the above mentioned distributions, we developed the following theorem:

Theorem 1. If the r -th raw moment (μ_r') for any continuous distribution with probability density function $f(x); -\infty < x < \infty$, exists, then the mgf of that distribution also exists.

Proof: If the mgf of a random variable X exists, then it is defined by

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)dx \\ &= \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x)dx \end{aligned}$$

Now, by definition

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx$$

So we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

i.e., the coefficient of $t^r/r\,!$ is the r -th raw moment of X . So we can say that for the existence of μ_r', there must exist mgf of X.

We now derive the mgfs of some distributions in the forms of some theorems.

Theorem 2. If the random variable X follows t-distribution with degrees of freedom n, then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^{2r}}{(2r)!} n^r \frac{\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \; ; n > 2r$$

Proof: By definition,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{dx}{\sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= C \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} \frac{dx}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$$

Where

$$C = \frac{1}{\sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right)}$$

Now, since for odd values of r, the value of the integral becomes zero, we replace r by 2 r , and thus get

$$M_X(t) = 2C\int_0^\infty \sum_{r=0}^\infty \frac{(tx)^{2r}}{(2r)!}\frac{dx}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= C\sum_{r=0}^\infty n^{r+\frac{1}{2}}\frac{t^{2r}}{(2r)!}\int_0^\infty \frac{\left(\frac{x^2}{n}\right)^{r-\frac{1}{2}} d\left(\frac{x^2}{n}\right)}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \sum_{r=0}^\infty \frac{t^{2r}}{(2r)1}n^r \frac{B\left(\frac{1}{2}+r,\frac{n}{2}-r\right)}{B\left(\frac{1}{2},\frac{n}{2}\right)} \text{ [puttirg the value of } C]$$

$$= \sum_{r=0}^\infty \frac{t^{2r}}{(2r)1}n^r \frac{\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\ ; n > 2r$$

This completes the proof.
Hence we get

$$\mu_{2\mathrm{r}} = n^r \frac{\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\ ; n > 2r$$

Theorem 3. If the random variable X follows F-distribution with n_1 and n_2 degrees of freedom, then its mgf is defined by

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\left(\frac{n_2}{n_1}\right)^r \frac{B\left(\frac{n_1}{2}+r,\frac{n_2}{2}-r\right)}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)};\quad n_2 > 2r$$

Proof: We have

$$M_X(t) = \int_0^\infty e^{tx} \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1} dx}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2} x\right)^{\frac{n_1+n_2}{2}}}$$

$$= C \int_0^\infty \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} \frac{x^{\frac{n_1}{2}-1} dx}{\left(1 + \frac{n_1}{n_2} x\right)^{\frac{n_1+n_2}{2}}}, \text{ where } C = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= C \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty \frac{x^{\frac{n_1}{2}+r-1} dx}{\left(1 + \frac{n_1}{n_2} x\right)^{\frac{n_1+n_2}{2}}}$$

$$= C \sum_{r=0}^{\infty} \frac{t^r}{r1} \left(\frac{n_2}{n_1}\right)^{\frac{n_1}{2}+\mathrm{r}} \int_0^\infty \frac{\left(\frac{n_1}{n_2} x\right)^{\frac{n_1}{2}+x-1} d\left(\frac{n_1}{n_2} x\right)}{\left(1 + \frac{n_1}{n_2} x\right)^{\frac{n_1+n_2}{2}}}$$

$$= C \sum_{\mathrm{r}=0}^{\infty} \frac{t^r}{r1} \left(\frac{n_2}{n_1}\right)^{\frac{n_1}{2}+\mathrm{r}} B\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right); n_2 > 2r$$

Putting the value of C, we get the required result.
Consequently,

$$\mu_r' = \left(\frac{n_2}{n_1}\right)^r \frac{B\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \; ; n_2 > 2r$$

Theorem 4. If the random variable X follows the Pareto distribution

$$f(x; k, \theta) = \frac{\theta k^\theta}{x^{\theta+1}}; k < x < \infty$$

then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\theta k^r}{(\theta - r)}$$

We omit the proof. Obviously,

$$\boldsymbol{\mu'_{\mathrm{r}}} = \frac{\boldsymbol{\theta k^r}}{\boldsymbol{\theta - r}}$$

Theorem 5. If a random variable X follows a non-central t distribution with degrees of freedom n and non-centrality parameter λ, then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^{2\mathrm{r}}}{(2r)!} \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k!} n^r \frac{B\left(r+k+\frac{1}{2}, \frac{n}{2} - x\right)}{B\left(\frac{1}{2}+k, \frac{n}{2}\right)} \; ; n > 2r$$

Proof: By definition,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k!} \frac{x^{2k}dx}{n^{\frac{1}{2}+k} B\left(\frac{1}{2}+k, \frac{n}{2}\right)\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}+k}}$$

$$= \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} \sum_{k=0}^{\infty} C_k \frac{x^{2k}dx}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}+k}} \text{, where } C_k = \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k1 n^{\frac{1}{2}+k} B\left(\frac{1}{2}+k, \frac{n}{2}\right)}$$

Now, since for odd values of r, the value of the integral becomes zero, we replace r by $2r$, and thus get

$$M_X(t) = 2\int_0^{\infty} \sum_{x=0}^{\infty} \frac{(tx)^{2r}}{2r!} \sum_{k=0}^{\infty} c_k \frac{x^{2k}dx}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}+k}}$$

$$= \sum_{r=0}^{\infty} \frac{(t)^{2\mathrm{r}}}{2r!} \sum_{k=0}^{\infty} c_k n^{r+k+\frac{1}{2}} \int_0^{\infty} \frac{\left(\frac{x^2}{n}\right)^{r+k-\frac{1}{2}} d\left(\frac{x^2}{n}\right)}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}+k}}$$

$$= \sum_{r=0}^{\infty} \frac{(t)^{2r}}{2r!} \sum_{k=0}^{\infty} c_k n^{r+k+\frac{1}{2}} B\left(r+k+\frac{1}{2}, \frac{n}{2} - r\right)$$

Putting the value c_k, we get the required result. Obviously,

$$\mu_r' = \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k!} n^r \frac{B\left(r+k+\frac{1}{2}, \frac{n}{2}-r\right)}{B\left(\frac{1}{2}+k, \frac{n}{2}\right)} \; ; n > 2r$$

Theorem 6. If a random variable X follows a non-central F distribution with degrees of freedom n_1 and n_2 and non-centrality parameter λ, then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left(\frac{n_2}{n_1}\right)^r \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k1} \frac{B\left(\frac{n_1}{2}+k+r, \frac{n_2}{2}-r\right)}{B\left(\frac{n_1}{2}+k, \frac{n_2}{2}\right)} \; ; n_2 > 2r$$

Proof: By definition,

$$M_X(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k!} \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}+k} x^{\frac{n_1}{2}+k-1} dx}{B\left(\frac{n_1}{2}+k, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}+k}}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{\Gamma!} \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k1} \left(\frac{n_2}{n_1}\right)^r \int_0^{\infty} \frac{\left(\frac{n_1}{n_2}x\right)^{\frac{n_1}{2}+k+\mathrm{r}-1} d\left(\frac{n_1}{n_2}x\right)}{B\left(\frac{n_1}{2}+k, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}+k}}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{\Gamma!}\left(\frac{n_2}{n_1}\right)^r \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k1} \frac{B\left(\frac{n_1}{2}+k+r, \frac{n_2}{2}-r\right)}{B\left(\frac{n_1}{2}+k, \frac{n_2}{2}\right)}$$

This completes the proof. Obviously,

$$\mu_r' = \left(\frac{n_2}{n_1}\right)^r \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^k}{k1} \frac{B\left(\frac{n_1}{2}+k+r, \frac{n_2}{2}-r\right)}{B\left(\frac{n_1}{2}+k, \frac{n_2}{2}\right)} \; ; n_2 > 2r$$

Theorem 7. If a random variable X follows the beta distribution

$$f(x; p, q) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}; 0 < x < 1$$

then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{B(p+r,q)}{B(p,q)}$$

We omit the proof. Clearly,

$$\mu_r' = \frac{B(p+r,q)}{B(p,q)}$$

Theorem 8. If the random variable X follows a lognormal distribution with parameters μ and σ^2, then its mgf is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r1} e^{\mu r + \frac{1}{2} r^2 \sigma^2}$$

We omit the proof. Obviously,

$$\mu_x' = e^{\mu r + \frac{1}{2} r^2 \sigma^2}$$

3. Conclusion

The results obtained in the paper in the form of theorems for some continuous distributions may be applied for finding moment generating function as well as moments of other distributions.

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